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THESIS

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to the
Problem of Replenishing Task Forces at Sea

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ABSTRACT

An analytical approach to the problem of replenishing task forces at sea is investigated, using a random walk model. The type of replenishment operations considered consists of two replenishment ships of different types, each replenishing L combatant ships of different types. The combatant ships are initially distributed into queues of length M and N between the two replenishment ships. As each combatant ship completes its first replenishment, it then enters the queue of the other replenishment ship. Thus each combatant ship is replenished by both replenishment ships. Replenishment times for the combatant ships are assumed to be independent, positive valued random variables whose distributions are dependent upon both replenishment ship type and combatant ship type. The replenishment of the task force is completed whenever 2L replenishments, two for each combatant ship, are completed. The total replenishment time is shown to be the maximum of certain partial sums of the individual replenishment times. A general expression for the distribution function of the total replenishment time is sought by various methods of analysis. A computer simulation is used to gain a better understanding of the form of the distribution function of total replenishment time for a specific underway replenishment operation. Some examples are discussed.

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I. THE UNDERWAY REPLENISHMENT OPERATION

The underway replenishment operation is the means by which needed supplies are provided to Navy fleet units while underway at sea. It provides these units with the capability of operating at great distances from Navy shore facilities, enables them to remain on station for extended periods, extends their period of effective combat readiness, and limits diversion from their assigned mission. The supplies provided may include, but need not be limited to, petroleum products, missiles, ammunitions, stores, and provisions. However, no matter what is to be transferred between the transferring replenishment ships and the receiving combatant ships, several points are apparent, the underway replenishment operation increases the vulnerability of the engaged units, restricts their mobility, and diverts the involved combatant ships from their principal missions. The Navy is continually striving for the development of techniques and methods which will reduce the limitations imposed on fleet operations by the necessity for underway replenishment. Principally, the Navy desires to reduce the vulnerability of the engaged units, increase the unit mobility while engaged, reduce the time required to complete replenishment, and limit interference with the assigned missions of the combatant units.

To attain these goals many improvements of a material nature have been accomplished. The introduction of faster replenishment ships, such as the combat stores ship (AFS), reduces the vulnerability and increases the mobility of the engaged units by allowing underway replenishments to be conducted at higher speeds than before. The

introduction of multi-commodity replenishment ships, such as the fast combat support ship (AOE) and the fleet replenishment tanker (AOR), reduces the total number of replenishments required since several needs can be filled during a single replenishment. The introduction of the helicopter as a transfer agent affords greater separation of task force units, reduces the demands for alongside replenishments, and minimizes interference with the assigned mission of the involved combatant ship. The introduction of new wire-highline transfer systems, such as the fast automatic schuttle transfer cargo handling system (FAST), provides a higher rate of cargo transfer and an increased load capacity, thereby reducing the required time alongside. In addition, internal cargo handling improvements in both the receiving and transferring ships have resulted in reduced breakout and strikedown times, thus again resulting in reduced replenishment times. All these improvements have had one element in common. Their accomplishment has resulted in a reduction in the actual replenishment time for an individual ship, and a subsequent reduction in the total replenishment time for the involved task force.

Improvements have not been limited to those of a material nature. The underway replenishment operation has also been viewed as a problem in operations analysis. The objective of such an approach is a better understanding of the underway replenishment operation as viewed in an analytical manner.

The underway replenishment operation can be viewed in many analytical contexts, but more readily lends itself to representation as a multi-server queueing model with a finite number of customers. The number of servers can be established by the number of replenishment ships involved,

while the number of customers can be established by the number of ships in the task force scheduled for underway replenishment operations. It should be noted that, under normal conditions, a definitive sequence of individual replenishments has been established prior to the commencement of the underway replenishment operation. This is done as a method of optimizing replenishment ship utilization, while hopefully minimizing total replenishment time for the task force. Thus, while the underway replenishment operation may be viewed in terms of a queueing problem, it has definite job-shop scheduling problem overtones.

One aspect of a typical queueing model must be ignored for the underway replenishment operation. It is normal to consider all service times to have come from the same probability distribution in a typical queueing model. Since many different types of replenishment and combatant ships are involved in a typical underway replenishment operation, this seems to be an unreasonable assumption for any realistic model of the operation. Analysis of actual underway replenishment data has supported this contention and has yielded approximations for the appropriate distribution functions for various typical underway replenishment operations.

II. THE UNDERWAY REPLENISHMENT MODEL

The basic underway replenishment model that was developed by Waggoner [1] and later expanded by Patterson [2] is the domain of inquiry for this thesis. This model assumes that two replenishment ships and L combatant ships are scheduled to conduct underway replenishment operations; a total of 2L replenishments are to be conducted, L by each replenishment ship. Prior to the commencement of the replenishment operation, the L combatant ships are given a specific replenishment sequence and assigned to queues of length M and N, where $M + N = L$, associated with the first and second replenishment ships, respectively. For each combatant ship this means that a definitive order of replenishment has been established and that each combatant ship has initially been assigned to one of the two queues. In addition, this means that the individual combatant ship both succeeds and precedes the same ships in each queue and that this queueing sequence is inalterable.

The movement of the combatant ships during the underway replenishment operation is envisioned as a sequence of steps in a random walk. This random walk is performed on an L x L grid on which the coordinates of each grid node represent a possible combination of the numbers of combatant ships replenished by each replenishment ship, i.e., the node (i,j) on the grid indicates that i combatants have been replenished by the first replenishment ship and j by the second replenishment ship. It is obvious from the replenishment sequence that certain grid nodes represent infeasible combinations; it is infeasible, for example, for the first replenishment ship to have completed $M + k$ replenishments unless the second replenishment ship has completed at least k, where

$k = 1,2,\ldots,N$. This infeasibility can be stated in constraint equations. If the abscissa, Z_1, of the grid denotes the number of replenishments completed by the first replenishment ship, while the ordinate, Z_2, of the grid denotes the number of replenishments completed by the second replenishment ship. The following equations constitute the necessary constraints:

$$Z_1 = 0$$
$$Z_1 = M + Z_2$$
$$Z_1 = L$$
$$Z_2 = 0$$
$$Z_2 = N + Z_1$$
$$Z_2 = L$$

All feasible nodes either lie on or within the boundaries established by the intersection of these constraint equations. Figure 1 depicts an L x L grid with the constraint equations superimposed.

In order to establish a common terminology for later discussion and investigation of the model, various definitions and theorems stated by Patterson [2], are presented.

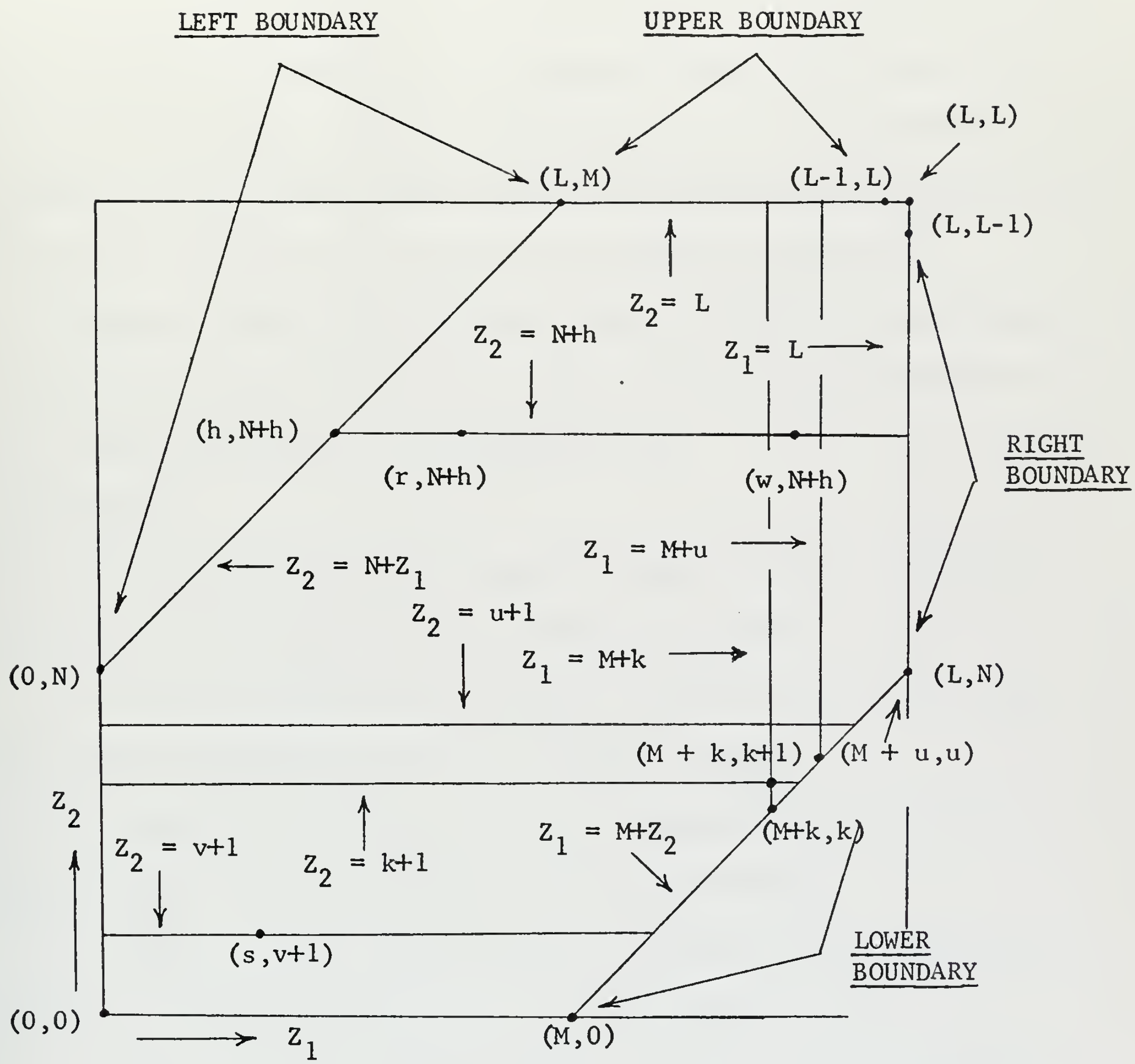
Definition 1

The CONSTRAINT BOUNDARY of the random walk grid is defined as follows (see Figure 1):

a. the UPPER BOUNDARY consists of those nodes along the line $Z_2 = L$ with abscissa k, where $M \leq k \leq L-1$;

b. the RIGHT BOUNDARY consists of those nodes along the line $Z_1 = L$ with ordinate h, where $N \leq h \leq L-1$;

c. the LOWER BOUNDARY consists of those nodes along the line $Z_1 = M + Z_2$, where $0 \leq Z_2 \leq N-1$; and



Underway Replenishment Model

Random Walk Grid

FIGURE 1

d. the LEFT BOUNDARY consists of those nodes along the line $Z_2 = N + Z_1$, where $0 \leq Z_1 \leq M-1$.

Definition 2

A STEP is a horizontal or vertical line between adjacent nodes of the grid; a PATH is a sequence of non-negative steps which starts at the origin and terminates at (L,L) and does not cross the constraint boundary; and X_i and Y_j denote the times required to complete the ith horizontal and jth vertical step, respectively, of any path.

From Definition 2 it is evident that X_i is the ith replenishment time for the first replenishment ship, while Y_j is the jth replenishment time for the second replenishment ship.

Definition 3

The set of all feasible paths from the origin to (L,L) is decomposed into four mutually exclusive and exhaustive subsets:

A: the set of all paths that pass through the node (L-1,L) and do not pass through any nodes on the lower boundary;

B: the set of all paths that pass through the node (L-1,L) and pass through at least one node on the lower boundary;

C: the set of all paths that pass through the node (L,L-1) and do not pass through any nodes on the left boundary; and

D: the set of all paths that pass through the node (L,L-1) and pass through at least one node on the left boundary.

In addition, subsets B and D can be further decomposed:

B_k: the subset of all paths in B which last pass through the lower boundary at node $(M + k,k)$ for $k = 0,1,\ldots,N-1$

D_h: the subset of all paths in D which last pass through the left boundary at node $(h, N+h)$ for $h = 0,1,\ldots, M-1$.

Theorem 1

The total replenishment time is the actual time required to complete a path through the grid and is equal to one of the following values:

a. the total length of time required for any path in A is

$$T_A = \sum_{i=1}^{L} X_i;$$

b. the total length of time required for any path in B_k is

$$T_B^{\,k} = \sum_{i=M+k+1}^{L} + \sum_{j=1}^{k+1} Y_i \quad \text{for } k = 0,1,\ldots,N-1;$$

c. the total length of time required for any path in C is

$$T_C = \sum_{j=1}^{L} Y_i; \text{ or}$$

d. the total length of time required for any path in D_h is

$$T_D^{\,h} = \sum_{i=1}^{h+1} X_i + \sum_{j=N+h+1}^{L} Y_i \quad \text{for } h = 0,1,\ldots,M-1.$$

Waggoner and Patterson both assumed that the X_i and Y_j were independent, random variables from the family of exponential probability distributions. Waggoner tacitly assumed that the replenishment rate was dependent only upon the replenishment ship type and that replenishment rates λ and μ could be used for all replenishments conducted by the first replenishment ship and the second replenishment ship, respectively. Patterson generalized this assumption to the extent that the replenishment rate was allowed to be dependent upon both the replenishment ship and the combatant ship types involved. Thus replenishment rates λ_i and μ_j were assumed for the ith replenishment by the first replenishment ship and the jth replenishment by the second replenishment ship, respectively. It should be noted here, however, that Patterson developed Theorem 1 without referral to his assumption

of exponentiality. Thus Theorem 1 holds without any assumption being made regarding the distribution of the individual replenishment times.

An interesting comparison can be made regarding the investigative approaches of Waggoner and Patterson. Waggoner approached the model by considering the first step taken and the subsequent sequence of steps of shortest duration taken during the random walk. Thus the exponentiality assumption, by means of the memoriless property, insured that the duration of each step continued to be exponentially distributed. This greatly simplified his derivation of the Laplace transform for the distribution function of the total replenishment time. On the other hand, Patterson reversed the sequence and considered the last step taken and the subsequent sequence of steps of longest duration in the reverse order. In this case, as noted above, the exponentiality assumption was not applied. Patterson, then using the exponentiality assumption, was able to develop the Laplace transform for the distribution function of total replenishment time only for the case of two combatant ships initially distributed one to each replenishment ship.

III. TOTAL REPLENISHMENT TIME

The previous section has intimated that total replenishment time can be one of certain feasible values. It was, in fact, stated that total replenishment time was either T_A, or $T_B{}^k$ (for some integer k where $0 \leq k \leq N-1$), or T_C, or $T_B{}^h$ (for some integer h where $0 \leq h \leq M-1$). In this section, by means of the following theorem and proof, it is shown that total replenishment time is actually the maximum of those feasible values.

<u>Theorem 2</u>

The total replenishment time, i.e., the time required to move from node (0,0) to node (L,L) of the random walk grid, is the maximum of all feasible values. Let T denote total replenishment time, then

$$T = \max\left(T_A;\ T_B{}^k \text{ for } k = 0,1,\ldots,N-1;\ T_C;\ T_D{}^h \text{ for } h = 0,1,\ldots,M-1\right).$$

<u>Proof</u>

a. Assume that the path taken was from A. Thus it must be shown that $T_A > T_B{}^k$ for $k = 0,1,\ldots,N-1$; $T_A > T_C$; and $T_A > T_D{}^h$ for $h = 0,1,\ldots,M-1$.

1) Because the path is in A, it must not touch the lower boundary. Thus it must reach the line $Z_2 = k + 1$ (see Figure 1) before it reaches the line $Z_1 = M + k$ for any $k = 0,1,\ldots,N-1$. This implies that

$$\sum_{i=1}^{M+k} X_i > \sum_{j=1}^{k+1} Y_j \text{ for any } k = 0,1,\ldots,N-1.$$

Now by adding the partial sum $\sum_{i=M+k+1}^{L} X_i$ to both sides of the above inequality, it is seen that $T_A > T_B{}^k$ for any $k = 0,1,\ldots,N-1$.

2) Because the path is in A, it must reach the upper boundary before it reaches the right boundary. This implies that

$$\sum_{i=1}^{L} X_i > \sum_{j=1}^{L} Y_j .$$

Therefore, $T_A > T_C$.

3) Let the path in A first touch the line $Z_2 = N + h$ (see Figure 1) at the node $(r,N+h)$ where $h \leq r \leq L-1$ for any $h = 0,1,\ldots,M-1$. Since the line $Z_2 = N + h$ must be reached vertically, the $(r + 1)$th horizontal step must have begun before the node $(r,N+h)$ is reached. The $(N+h+1)$th vertical step is either just beginning $(h+1 \leq r \leq L-1)$ at node $(r,N+h)$ or will begin on completion of the $(r + 1)$th horizontal step ($r = h$). Since the upper boundary must be reached before the right boundary, this implies that either

$$\sum_{i=r+1}^{L} X_i > \sum_{j=N+h+1}^{L} Y_j \text{ if } h+1 \leq r \leq L-1, \text{ or}$$

$$\sum_{i=r+2}^{L} X_i > \sum_{j=N+h+1}^{L} Y_j \text{ if } r = h.$$

In either case, the above inequalities imply that

$$\sum_{i=h+2}^{L} X_i > \sum_{j=N+h+1}^{L} Y_j \text{ for any } h = 0,1,\ldots,M-1.$$

Now if the partial sum $\sum_{i=1}^{h+1} X_i$ is added to both sides of the last inequality, it can be seen that $T_A > T_D^{\,h}$ for any $h = 0,1,\ldots,M-1$.

b. Assume that the actual path taken was from B_k where k is fixed and $0 \leq k \leq N-1$. Thus it must be shown that $T_B^{\,k} > T_A$; $T_B^{\,k} > T_B^{\,n}$ for $n \neq k$, $0 \leq n \leq N-1$; $T_B^{\,k} > T_C$; and $T_B^{\,k} > T_D^{\,h}$ for $h = 0,1,\ldots,M-1$.

1) Because the path is in B_k, it must reach the line $Z_1 = M + k$ (see Figure 1) before reaching the $Z_2 = k + 1$. Thus

$$\sum_{j=1}^{k+1} Y_j > \sum_{i=1}^{M+k} X_i .$$

Now by adding the partial sum $\sum_{i=M+k+1}^{L} X_i$ to both sides of the above inequality, it can be seen that $T_B^k > T_A$.

2) Because the path is in B_k, it must pass through the node $(M+k,k)$. Thus the $(M+k+1)$th horizontal step begins at the same time as the $(k+2)$th vertical step. Since the path cannot pass through any node $(M+u,u)$ where $k < u \leq N-1$, it must reach the line $Z_2 = u + 1$ (see Figure 1) before reaching the line $Z_1 = M+u$. These conditions imply that

$$\sum_{i=M+k+1}^{M+u} X_i > \sum_{j=k+2}^{u+1} Y_j, \text{ for } k < u \leq N-1.$$

Now by adding the two partial sums $\sum_{i=M+u+1}^{L} X_i$ and $\sum_{j=1}^{k+1} Y_j$ to both sides of the above inequality, it can be seen that $T_B^k > T_B^u$ for $k < u \leq N-1$.

3) Let the path in B_k first touch the line $Z_2 = v+1$ (see Figure 1) at the node $(s,v+1)$ where $0 \leq s \leq M+v$ and $0 \leq v < k$. The $(v+2)$th vertical step begins at this node, while the $(s+1)$th horizontal step either just begins $(s = M+v)$ or has already begun $(0 \leq s \leq M+v-1)$. Since it is necessary that the line $Z_2 = k+1$ (see Figure 1) not be reached until after the line $Z_1 = M + k$ is reached, this implies that either

$$\sum_{j=v+2}^{k+1} Y_j > \sum_{i=s+1}^{M+k} X_i \text{ if } s = M + v \text{ or}$$

$$\sum_{j=v+2}^{k+1} Y_j > \sum_{i=s+2}^{M+k} X_i \text{ if } 0 \leq s \leq M + v-1.$$

In either case, this implies that

$$\sum_{j=v+2}^{k+1} Y_j > \sum_{i=M+v+1}^{M+k} X_i \text{ for } 0 \leq v < k.$$

Now if the partial sums $\sum_{j=1}^{v+1} Y_j$ and $\sum_{i=M+k+1}^{L} X_i$ are added to both sides of the last inequality, it can be seen that $T_B^k > T_B^v$ for $0 \leq v < k$.

4) Because the path is in B_k, the (k+2)th vertical step and the (M+k+1)th horizontal step begin at the same time. Since the upper boundary must be reached before the right boundary, this implies that

$$\sum_{k=M+k+1}^{L} X_i > \sum_{j=k+2}^{L} Y_j .$$

Now by adding the partial sum $\sum_{j=1}^{k+1} Y_j$ to both sides of the above inequality, it can be seen that $T_B^k > T_C$.

5) Because the path is in B_k, it cannot reach the line $Z_2 = k+1$ (see Figure 1) before reaching the line $Z_1 = M+k$, which implies that

$$\sum_{j=1}^{k+1} Y_j > \sum_{i=1}^{M+k} X_i .$$

This in turn implies that

$$(1) \quad \sum_{j=1}^{k+1} Y_j > \sum_{i=1}^{h+1} X_i \text{ for any } h = 0,1,\ldots,M-1.$$

Further, let the path in B_k first reach the line $Z_2 = N + h$ (see Figure 1) for any $h = 0,1,\ldots,M-1$ at some node (w,N+h) where $M+k \leq w < L$. Since the path reaches $Z_2 = N + h$ through a vertical step, the (N+h+1)th step is just beginning at node (w,N+h) while the (w+1)th horizontal step has already begun. Since the upper boundary must be reached before the right boundary, it is necessary that

$$\sum_{i=w+1}^{L} X_i > \sum_{j=N+h+1}^{L} Y_j \text{ for } M + k \leq w < L.$$

This in turn implies that

$$(2) \quad \sum_{i=M+k+1}^{L} X_i > \sum_{j=N+h+1}^{L} Y_j \text{ for any } h = 0,1,\ldots,M-1.$$

Now if the inequalities (1) and (2) above are added together, it can be seen that $T_B^k > T_D^h$ for $h = 0,1,\ldots,M-1$.

c. If the path taken was in C, T_C is larger than the other feasible values. This is proven analogously to part a.

d. If the path taken was in D_h, for some $h = 0,1,\ldots,M-1$, T_D^h is larger than the other feasible values. This is proven analogously to part b.

This concludes the proof of Theorem 2.

By way of illustrating the result of Theorem 2, an example where constant individual replenishment times are involved is discussed. It is assumed that $X_i = A$ and $Y_j = B$ for $i,j = 1,2,\ldots,L$ for the general case of L combatant ships, distributed into queues of length M and N, associated with the first and second replenishment ships, respectively. If neither $N = 0$ nor $M = 0$, the following are the feasible values for T:

$$T_A = LA$$
$$T_B^k = (N-k)A + (k+1)B \text{ for } k = 0,1,\ldots,N-1$$
$$T_C = LB$$
$$T_D^h = (h+1)A + (M-h)B \text{ for } h = 0,1,\ldots,M-1.$$

T, the maximum of all feasible values, is clearly equal to either LA or LB, depending upon the values of A and B. If $A \geq B$, then $T = LA$; if $B > A$, then $T = LB$. If $N = 0$, all feasible paths are in D and the following are the feasible values of T:

$$T_D^h = (h+1)A + (L-h)B \text{ for } h = 0,1,\ldots,L-1.$$

T is clearly equal to either $(A + LB)$ or $(LA + B)$, depending upon the values of A and B. If $A \geq B$, then $T = (LA + B)$; if $B > A$, then $T = (A + LB)$. If $M = 0$, the preceding argument holds if the set D is replaced by the set B.

IV. IDENTIFICATION OF THE GENERAL PROBLEM

In the preceding section it has been shown that the actual total replenishment time, denoted T, is the maximum of all feasible values. Thus, while the exact composition of T as expressed in terms of the individual replenishment times is known, the general form of its distribution function as expressed in terms of those for the individual replenishment times remains unidentified. In regard to this last statement, in this or any succeeding section the following notation has been used: the distribution function of X_i is denoted F_i, while that of the Y_j is denoted G_j; if densities are assumed, they will be denoted by the appropriate small letter with subscript.

The composition of T and the distribution function of T when $L \leq 2$, while of little practical interest, are presented below as an introduction to the analytical techniques used in the attempt to identify the general distribution function of T, when $L \geq 3$:

a. if $L = 1$ with $M = 1$ and $N = 0$, then $T = X_1 + Y_1$ and its distribution function is the convolution of F_1 and G_1;

b. if $L = 2$ with $M = 2$ and $N = 0$, then $T = X_1 + \max(X_2, Y_1) + Y_2$ and its distribution function is the convolution of F_1, H_{21}, and G_2, where H_{21} denotes the distribution function of $\max(X_2, Y_1)$, i.e., $F_2 G_1$;

c. if $L = 2$ with $M = 1$ and $N = 1$, then $T = \max(X_1, Y_1) + \max(X_2, Y_2)$ and its distribution function is the convolution of H_{11} and H_{22}, where H_{ii} denotes the distribution function of $\max(X_i, Y_i)$, i.e., $F_i G_i$ for $i = 1, 2$.

It should be noted that for all cases investigated, either above or beyond this point, an assumption that $M \geq N$ has been made. The symmetrical nature of the model has insured that any results obtained here are valid upon the interchange of M with N, X_i with Y_j, and F_i with G_j for the case when $M < N$.

The general distribution function of T can be represented by the following probability statements:

a. if $N = 0$, then

$$P(T \leq t) = P\left(\sum_{i=1}^{h+1} X_i + \sum_{j=h+1}^{L} Y_j \leq t \text{ for } h = 0,1,\ldots,L-1 \right);$$

b. if $N \geq 1$, then

$$P(T \leq t) = P\left(\sum_{i=1}^{L} X_i \leq t; \quad \sum_{i=M+k+1}^{L} X_i + \sum_{j=1}^{k+1} \leq t \text{ for } k = 0,1,\ldots,N-1; \right.$$

$$\left. \sum_{j=1}^{L} Y_j \leq t; \quad \sum_{i=1}^{h+1} X_i + \sum_{j=N+h+1}^{L} \leq t \text{ for } h = 0,1,\ldots,M-1 \right).$$

Viewing the summation terms involved in the two probability statements as random variables, denoted as T_n, it can be noted that when $N = 0$ L such random variables are present, while if $N \geq 1$ $L + 2$ such random variables are present. Since it was desired to use the technique of transformation of variables to simplify these probability statements, it was obvious that additional random variables had to be introduced in order that the one for one transformation demanded by the technique could be attained. Since the transformation was to be from the X_i and the Y_j into the T_n, L artificial random variables would have to be introduced if $N = 0$, while $L - 2$ would suffice if $N \geq 1$. These artificial random variables were selected to fulfill two purposes: primarily, they insure a one-to-one transformation, while secondarily,

they have no effect on the appropriate probability statements. For the case of $N = 0$, random variables of the form $\sum_{i=m}^{L} X_i$ for $m = 1,2,\ldots,L$ are introduced. These summations are specifically less than $\sum_{i=1}^{L} X_i + Y_L$. For the case of $N \geq 1$ random variables of the form $\sum_{i=m}^{L} X_i$ for $m = 2,3,\ldots, M$ and $\sum_{j=n}^{L} Y_j$ for $n = 2,3,\ldots, N$ are introduced. These summations are specifically less than $\sum_{i=1}^{L} X_i$ and $\sum_{j=1}^{L} Y_j$ respectively. Thus for either case the introduction of these particular summations into the appropriate probability statement have no effect.

The whole set of new random variables is defined below:

a. if $N = 0$,

$$T_m = \sum_{i=m}^{L} X_i \text{ for } m = 1,2,\ldots,L$$

$$T_{L+h+1} = \sum_{i=1}^{h+1} X_k + \sum_{j=h+1}^{L} Y_j \text{ for } h = 0,1,\ldots,L-1;$$

b. if $N \geq 1$,

$$T_m = \sum_{i=m}^{L} X_i \text{ for } m = 1,2,\ldots,M$$

$$T_{M+k+1} = \sum_{i=M+k+1}^{L} X_i + \sum_{j=1}^{k+1} Y_j \text{ for } k = 0,1,\ldots N-1$$

$$T_{L+n} = \sum_{j=n}^{L} Y_j \text{ for } n = 1,2,\ldots,N$$

$$T_{L+N+h+1} = \sum_{i=1}^{h+1} X_i + \sum_{j=N+h+1}^{L} Y_j \text{ for } h = 0,1,\ldots,M-1.$$

The Jacobian matrices of these transformations are shown in Figures 2, 3 and 4. The value of their determinants is the subject of Lemma 1.

Lemma 1

The Jacobian determinant of the transformation matrices shown in Figures 2, 3 and 4 is equal to 1, unless $L = 2$ with $M = 1$ and $N = 1$, i.e., if

JACOBIAN MATRIX OF T TRANSFORMATIONS WHEN N = 0

	X_1	X_2	...	X_{L-1}	X_L	Y_1	Y_2	...	Y_{L-1}	Y_L
T_1	1	1	...	1	1	0	0	...	0	0
T_2	0	1	...	1	1	0	0	...	0	0
.	.	.	...	.	.	.	.	...	.	.
.	.	.	...	.	.	.	.	...	.	.
.	.	.	...	.	.	.	.	...	.	.
T_L	0	0	...	0	1	0	0	...	0	0
T_{L+1}	1	0	...	0	0	1	1	...	1	1
.	.	.	...	.	.	.	.	...	.	.
.	.	.	...	.	.	.	.	...	.	.
.	.	.	...	.	.	.	.	...	.	.
T_{2L}	1	1	...	1	1	0	0	...	0	1

FIGURE 2

JACOBIAN MATRIX OF T TRANSFORMATIONS WHEN N = 1

	X_1	X_2	...	Y_{M-1}	X_M	X_L	Y_1	Y_2	...	Y_M	Y_L
T_1	1	1	...	1	1	1	0	0	...	0	0
T_2	0	1	...	1	1	1	0	0	...	0	0
.	.	.	...	.	.	.	.	.	...	.	.
.	.	.	...	.	.	.	.	.	...	.	.
.	.	.	...	.	.	.	.	.	...	.	.
T_M	0	0	...	0	1	1	0	0	...	0	0
T_L	0	0	...	0	0	1	1	0	...	0	0
T_{L+1}	0	0	...	0	0	0	1	1	...	1	1
T_{L+2}	1	0	...	0	0	0	0	1	...	1	1
.	.	.	...	.	.	.	.	.	...	.	.
.	.	.	...	.	.	.	.	.	...	.	.
.	.	.	...	.	.	.	.	.	...	.	.
T_{2L}	1	1	...	1	1	0	0	0	...	0	1

FIGURE 3

<u>JACOBIAN MATRIX OF T TRANSFORMATIONS WHEN $N \geq 2$</u>

	X_1	X_2	...	X_{M-1}	X_M	X_{M+1}	...	X_{L-1}	X_L	Y_1	Y_2	...	Y_{N-1}	Y_N	Y_{N+1}	...	Y_{L-1}	Y_L
T_1	1	1	...	1	1	1	...	1	1	0	0	...	0	0	0	...	0	0
T_2	0	1	...	1	1	1	...	1	1	0	0	...	0	0	0	...	0	0
.	.	.	...	.	.	.	...	.	.	.	.	...	.	.	.	...	.	.
.	.	.	...	.	.	.	...	.	.	.	.	...	.	.	.	...	.	.
.	.	.	...	.	.	.	...	.	.	.	.	...	.	.	.	...	.	.
T_M	0	0	...	0	1	1	...	1	1	0	0	...	0	0	0	...	0	0
T_{M+1}	0	0	...	0	0	1	...	1	1	1	0	...	0	0	0	...	0	0
.	.	.	...	.	.	.	...	.	.	.	.	...	.	.	.	...	.	.
.	.	.	...	.	.	.	...	.	.	.	.	...	.	.	.	...	.	.
.	.	.	...	.	.	.	...	.	.	.	.	...	.	.	.	...	.	.
T_L	0	0	...	0	0	0	...	0	1	1	1	...	1	1	0	...	0	0
T_{L+1}	0	0	...	0	0	0	...	0	0	1	1	...	1	1	1	...	1	1
T_{L+2}	0	0	...	0	0	0	...	0	0	0	1	...	1	1	1	...	1	1
.	.	.	...	.	.	.	...	.	.	.	.	...	.	.	.	...	.	.
.	.	.	...	.	.	.	...	.	.	.	.	...	.	.	.	...	.	.
.	.	.	...	.	.	.	...	.	.	.	.	...	.	.	.	...	.	.
T_{L+N}	0	0	...	0	0	0	...	0	0	0	0	...	0	1	1	...	1	1
T_{L+N-1}	1	0	...	0	0	0	...	0	0	0	0	...	0	0	1	...	1	1
.	.	.	...	.	.	.	...	.	.	.	.	...	.	.	.	...	.	.
.	.	.	...	.	.	.	...	.	.	.	.	...	.	.	.	...	.	.
.	.	.	...	.	.	.	...	.	.	.	.	...	.	.	.	...	.	.
T_{2L}	1	1	...	1	1	0	...	0	0	0	0	...	0	0	0	...	0	1

FIGURE 4

$|J|$ denotes the determinant of the Jacobian transformation matrix J, then $|J| = 1$.

Proof

a. Rows 1 through L + N have all 0's to the left of the main diagonal elements and the main diagonal elements are equal to 1.

b. Rows L + N + 1 through 2L-1 can be reduced to similar form by the application of the formula

$$R'_{L+N+i} = R_{L+N+i} - R_1 + R_{i+1} \quad \text{for } i = 1,2,\ldots,M-1,$$

where R_i denotes the ith row before application of the formula, and R'_i afterwards.

c. Row 2L can be reduced to a similar form by application of the formula

$$R'_{2L} = R_{2L} - R_1 \quad \text{if } N = 0,$$

$$R'_{2L} = R_{2L} - R_1 + R_{M+1} - R_{L+1} + R_{L+2} - R_1 + R_2 \quad \text{if } N = 1, \text{ or}$$

$$R'_{2L} = R_{2L} - R_1 + R_{M+1} - R_{L+1} + R_{L+2} \quad \text{if } N \geq 2.$$

The case for L = 2 with M = 1 and N = 1 was specifically excluded from Lemma 1 since the determinant of its Jacobian transformation matrix equals zero. This results from the inherent dependence of the involved summations. It is noted that this is the only case in which no artificial random variables can be introduced. Of course, the distribution function of T in this case was given earlier in this section.

A second transformation of variables was made at this point to gain further simplification. The following random variables were defined:

$$V_m = T_m - T_{m+1} \quad \text{for } m = 1,2,\ldots,2L-1$$

$$V_{2L} = T_{2L}.$$

The determinant of the resulting Jacobian transformation matrix equals 1 since the matrix is triangular with all main diagonal elements equal to 1, i.e., all elements below the main diagonal are 0's.

At this point it was decided to assume that the distribution functions of the individual replenishment times were continuous and possessed non-vanishing density functions when the X_i or Y_j are greater than 0. The joint density function of the X_i and the Y_j can thus be written

$$f(x_1,\ldots,x_L,y_1,\ldots,y_L) = \begin{cases} \prod_{i=1}^{L} f_i(x_i) \prod_{j=1}^{L} g_j(y_j) & x_i > 0 \text{ and } y_j > 0 \\ & \text{for } i,j=1,2,\ldots,L \\ 0 & \text{otherwise,} \end{cases}$$

since it was previously assumed that the X_i and the Y_j were independent random variables.

Since the appropriate Jacobian determinants are equal to 1, the joint density function of the V_n can also be written in the same manner

$$g(v_1,v_2,\ldots,v_{2L}) = \begin{cases} \prod_{i=1}^{L} f_i(x_i) \prod_{j=1}^{L} g_j(y_j) & x_i > 0 \text{ and } y_j > 0 \\ & \text{for } i,j=1,2,\ldots,L \\ 0 & \text{otherwise.} \end{cases}$$

The region in V-space for which $g(v_1,v_2,\ldots,v_{2L})$ does not vanish can be represented in terms of the V_m if the transformations relating the X_i and the Y_j to the V_m are used. These are given below:

a. if $N = 0$,

$$X_i = V_i \quad \text{for } i = 1,2,\ldots,L-1$$

$$X_L = \sum_{i=L}^{2L} V_i$$

$$Y_j = V_{j+1} + V_{L+j} \quad \text{for } j = 1,2,\ldots,L-2$$

$$Y_{L-1} = \sum_{i=L}^{2L} V_i + V_{2L-1}$$

$$Y_L = - \sum_{i=1}^{2L-1} V_i$$

b. if $N = 1$,

$$X_i = V_i \quad \text{for } i = 1,2,\ldots,M-1$$

$$X_M = V_1 + V_M + V_{L+1}$$

$$X_L = - V_1 + V_L + \sum_{i=L+2}^{2L} V_i$$

$$Y_j = V_j + V_{L+j} \quad \text{for } j = 1,2,\ldots,L-2$$

$$Y_{L-1} = V_1 + V_M + V_{L+1} + V_{2L-1}$$

$$Y_L = - 2V_1 - \sum_{i=2}^{L-1} V_i - V_{L+1} + V_{2L}$$

c. if $N \geq 2$,

$$X_i = V_i \quad \text{for } i = 1,2,\ldots,M-1$$

$$X_{M+k} = V_{M+k} + V_{L+k+1} \quad \text{for } k = 0,1,\ldots,N-2$$

$$X_{L-1} = V_1 + V_{L-1} + V_{L+N}$$

$$X_L = - V_1 + V_L + \sum_{i=L+N+1}^{2L} V_i$$

$$Y_j = V_{L+j} \quad \text{for } j = 1,2,\ldots,N-1$$

$$Y_{N+h} = V_{h+1} + V_{L+N+h} \quad \text{for } h = 0,1,\ldots,M-2$$

$$Y_{L-1} = V_M + V_{L+1} + V_{2L-1}$$

$$Y_L = - \sum_{i=1}^{M} V_i - V_{L+1} + V_{2L}$$

The above statements thus allow the original probability statements for the distribution function of T to be represented in terms

of the V_m in the following form:

$$P(T \leq t) = P\left(\sum_{i=m}^{2L} V_i \leq t \text{ for } m = 1,2,\ldots,2L \right)$$

provided that the joint density function for the V_m does not vanish. As an example, the joint density function of the V_m, expressed wholly in terms of the V_m, is presented below for the case $N \geq 2$:

$$g(v_1,v_2,\ldots,v_{2L}) = \begin{cases} \prod_{i=1}^{M-1} f_i(v_i) \prod_{i=M}^{L-2} f_i(v_i+v_{N+i+1}) f_{L-1}(v_i+v_{L-1}+v_{L+N}) \\ f_L(-v_1+v_L+\sum_{i=L+N+1}^{2L} v_i) \prod_{j=1}^{N-1} g_j(v_{L+j}) \prod_{j=N}^{L-2} g_j(v_{j+1-N}+v_{L+j}) \\ g_{L-1}(v_M+v_{L+1}+v_{2L-1}) g_L(-\sum_{i=1}^{M} v_i - v_{L+1}+v_{2L}) \\ \text{if condition (1) below exists} \\ 0 \qquad \text{otherwise.} \end{cases}$$

Condition (1) is expressed below:

$$v_i > 0 \quad i = 1,\ldots,M-1$$

$$v_i + v_{N+i+1} > 0 \quad i = M,\ldots,L-2$$

$$v_1 + v_{L-1} + v_{L+N} > 0$$

$$-v_1 + v_L + \sum_{i=L+N+1}^{2L} v_i > 0$$

$$v_{L+j} > 0 \quad j = 1,\ldots,N-1$$

$$v_{j+1-N} + v_{L+j} > 0 \quad j = N,\ldots,L-2$$

$$v_M + v_{L+1} + v_{2L-1} > 0$$

$$-\sum_{i=1}^{M} v_i - v_{L+1} + v_{2L} > 0.$$

It is evident that the above discussion has not concluded in a truly desirable result, namely a general form of the distribution

function of T that would facilitate ease of computation. The resulting forms, those containing the T_n and the V_m, do, however, exhibit properties that may prove useful in future attempts in this area. As the matter now stands, not even a relatively simple problem, such as that when L = 3 with M = 2 and N = 1, can be directly resolved. Appendix A depicts this particular problem as an illustration of the general difficulties encountered. In retrospect, it can be stated that the dependence relationships between the T_n, or those between the V_m, impose restrictions that cannot be overcome without some convenient manner of handling the joint distribution function of dependent random variables, such as that developed in this section.

V. A SIMULATION OF THE UNDERWAY REPLENISHMENT OPERATION

The preceding section, in which analytical methods were used, failed to produce the distribution function of T in a form feasible for specific computations. It was therefore decided to study the distribution of T by computer generation of T realizations for a specific case of underway replenishment. The primary intent of such a study was to gain a better understanding of the form of the distribution function of T for typical underway replenishment operations.

The general method used made direct use of the results of Theorem 2, i.e., that T is the maximum of all feasible values. The underway replenishment operation was not directly simulated. Instead, the individual replenishment times were simulated and the resultant values of T calculated, using the results of Theorem 2.

For the specific underway replenishment operation, it was decided to use an attack carrier task force replenishment operation. The task force was selected to be composed of eight combatant ships, an attack aircraft carrier (CVA), a light guided missile cruiser (CLG), two guided missile frigates (DLG), and four destroyers (DD). The two ship replenishment group was selected to be composed of a fleet oiler (AO) and an ammunition ship (AE). This particular selection was made because approximations for the distribution functions of the individual replenishment times for the above type ships were presented by Besecker in [3]. Thus Erlang distribution functions were assumed for the individual replenishment times; the specific gamma parameters, denoted k and λ, are shown in Table I. The general form of the Erlang distribution function is given in Appendix B.

PARAMETERS FOR ERLANG DISTRIBUTION FUNCTIONS

FOR

INDIVIDUAL REPLENISHMENT TIMES

		REPLENISHMENT SHIPS			
		AO		AE	
		k	λ	k	λ
COMBATANT SHIPS	CVA	11	0.09	3	0.03
	CLG	7	0.11	2	0.02
	DLG	7	0.11	2	0.02
	DD	4	0.05	3	0.06

TABLE I

In accordance with the model, the AO and AE were designated the first and second replenishment ships, respectively. The AO is then associated with the X_i, while the AE is associated with the Y_j. It was decided to establish a constant replenishment sequence at the AO. This sequence is CVA, DLG, DD, DD, CLG, DLG, DD, and DD. The replenishment sequence at the AE would depend upon the M and N combination being used for the generation of a T realization. The feasible sequences are shown in Table II.

The individual replenishment times were simulated by the inverse transformation method discussed in Appendix B. The property that an Erlang random variable is the sum of k random variables, each exponentially distributed with parameter λ, facilitates this method. To

insure that the random variates generated were from the assigned distributions, chi-square goodness of fit tests were made at the 0.01 level of significance. This method is discussed in Appendix B. For all 16 distributions of the individual replenishment times, the null hypotheses that the generated random variates came from the appropriate distributions were of course accepted.

REPLENISHMENT SEQUENCE AT AE									
M	8	7	6	5	4	3	2	1	0
N	0	1	2	3	4	5	6	7	8
REPLENISHMENT SEQUENCE	CVA	DD	DD	DLG	CLG	DD	DD	DLG	CVA
	DLG	CVA	DD	DD	DLG	CLG	DD	DD	DLG
	DD	DLG	CVA	DD	DD	DLG	CLG	DD	DD
	DD	DD	DLG	CVA	DD	DD	DLG	CLG	DD
	CLG	DD	DD	DLG	CVA	DD	DD	DLG	CLG
	DLG	CLG	DD	DD	DLG	CVA	DD	DD	DLG
	DD	DLG	CLG	DD	DD	DLG	CVA	DD	DD
	DD	DD	DLG	CLG	DD	DD	DLG	CVA	DD

TABLE II

A special feature was provided in the implementation of the computer program to enable study of the dependence of T on the M and N combination being used. The program provided that upon the nth generation of a T realization for a particular M and N combination, all ships would have the same individual replenishment times at both replenishment ships as for any other M and N combination, i.e., on the 50th generation of a T realization for M = 3 and N = 5, the CLG would

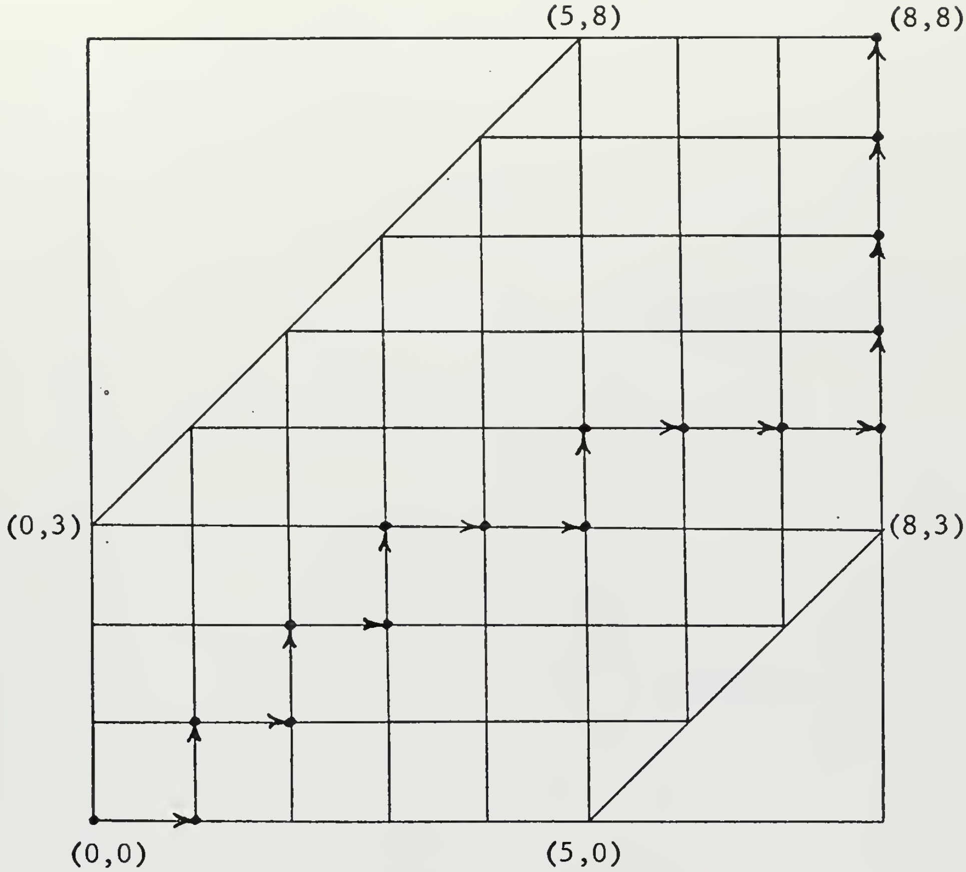
have the same replenishment times at both the AO and AE as for the 50th generation of a T realization for any other M and N combination of the combatant ships.

Five hundred T realizations were generated for each of the nine feasible M and N combinations. The random walk for a typical T realization is shown in Figure 5 as a random walk within the constraint boundary of the L x L grid and in Figure 6 as a random walk in two-dimensional time space. The T realization obtained by performing the random walk agrees with that obtained through an application of Theorem 2.

For each M and N combination, the generated data were used to construct frequency histograms of T realizations. Figure 7 is a typical frequency histogram. From these histograms, one basic observation could be made: the frequency histograms showed definite positive skewness, i.e., they were skewed to the right. The obvious contention at this point was that the distribution function had the same characteristic shape as did those of the individual replenishment times, i.e., the shape of a gamma distribution function.

A second observation could be made once the T sample means for the various M and N combinations were calculated. The results are similar to those cited by Milch and Waggoner in [4] for the expected values of T when the X_i and Y_j are exponentially distributed with parameters λ and μ, respectively. The minimal T sample mean occurred when M and N were equal and a range of M and N combinations exists in which there is only a slight variation of the T sample mean values. The T sample means are plotted against M and N combinations in Figure 8.

It was decided to investigate the contention that the distribution function of T was from the family of gamma distributions. Estimates of



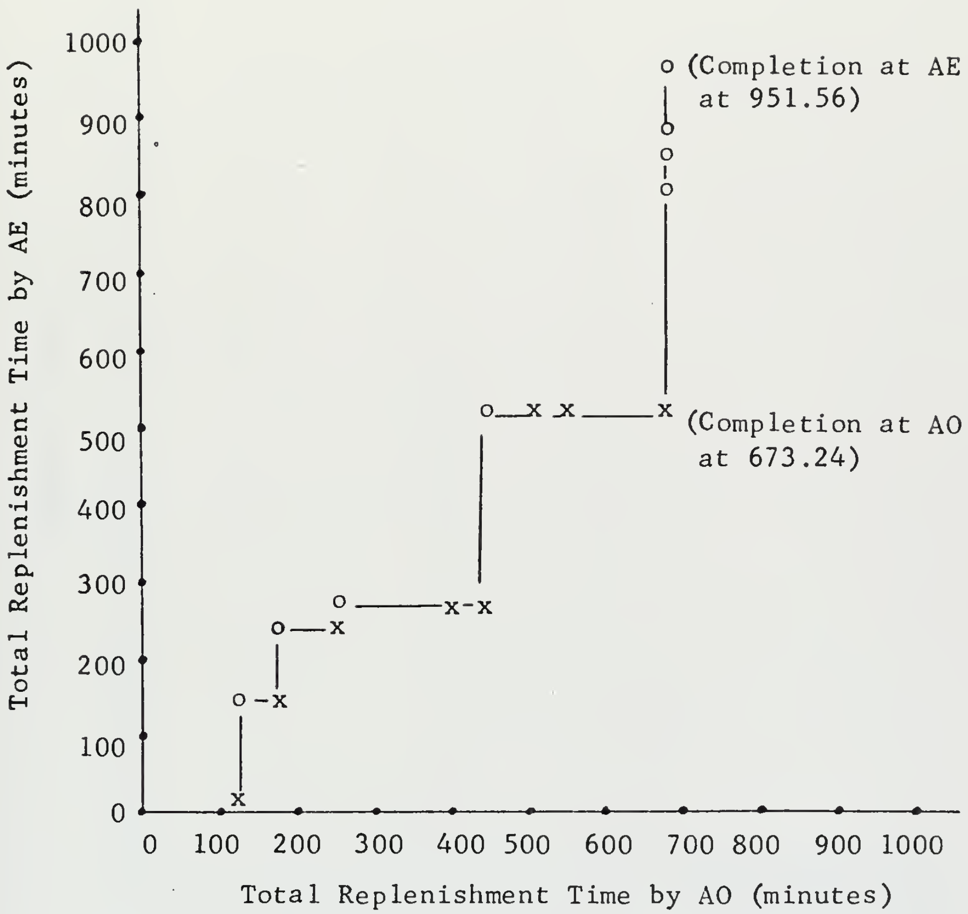
Individual Replenishment Times

X_1	= 123.35	Y_1	= 150.71
X_2	= 50.14	Y_2	= 88.46
X_3	= 81.12	Y_3	= 25.01
X_4	= 146.83	Y_4	= 257.16
X_5	= 34.64	Y_5	= 282.34
X_6	= 91.59	Y_6	= 48.44
X_7	= 21.43	Y_7	= 8.52
X_8	= 124.14	Y_8	= 90.92

$T = 951.56$

A Typical Random Walk for the
Specified Underway Replenishment Operation when $M = 5$ and $N = 3$

FIGURE 5



x: a completion of an individual replenishment by AO

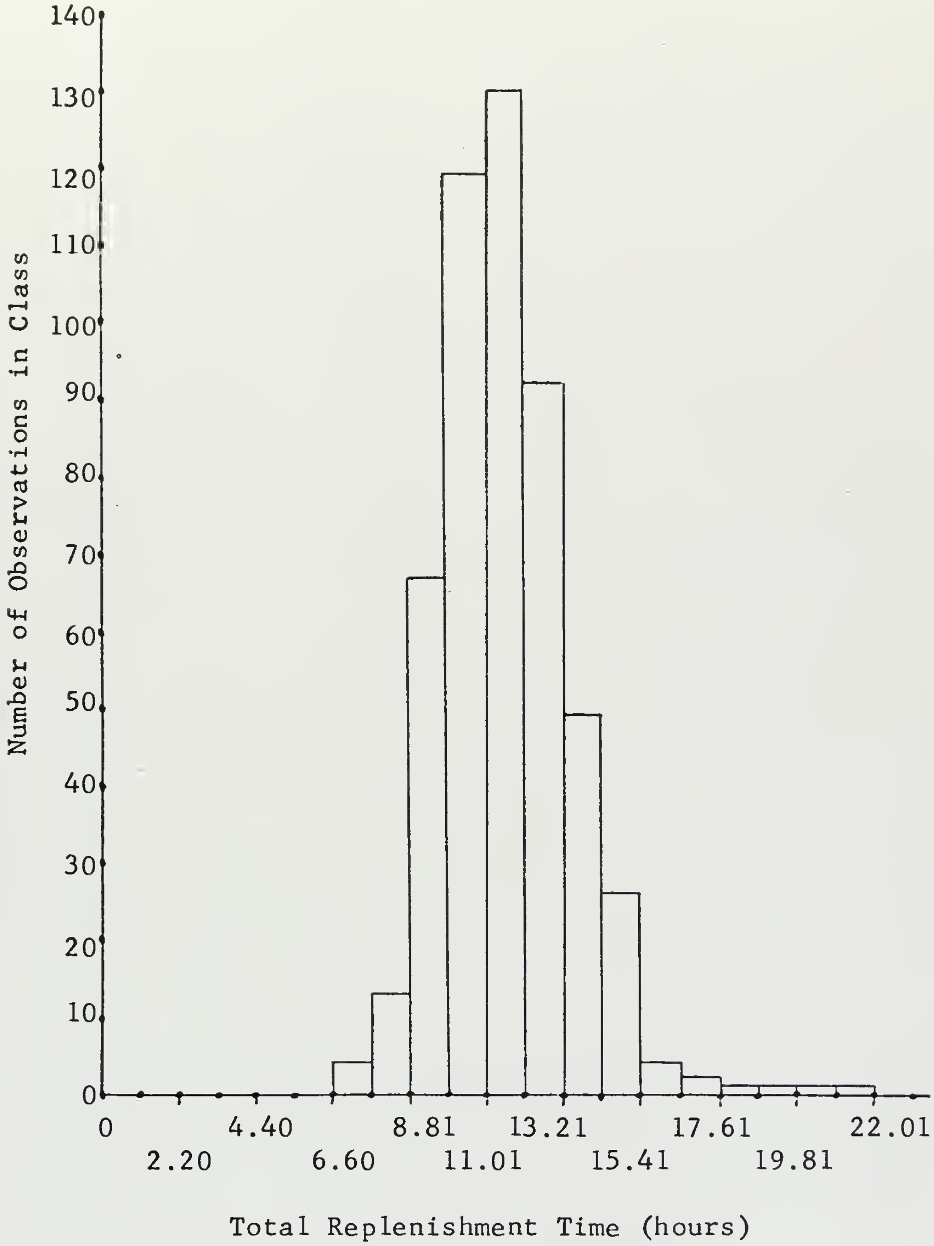
o: a completion of an individual replenishment by AE

<u>Note</u>

Data plotted is the same as used for Figure 5

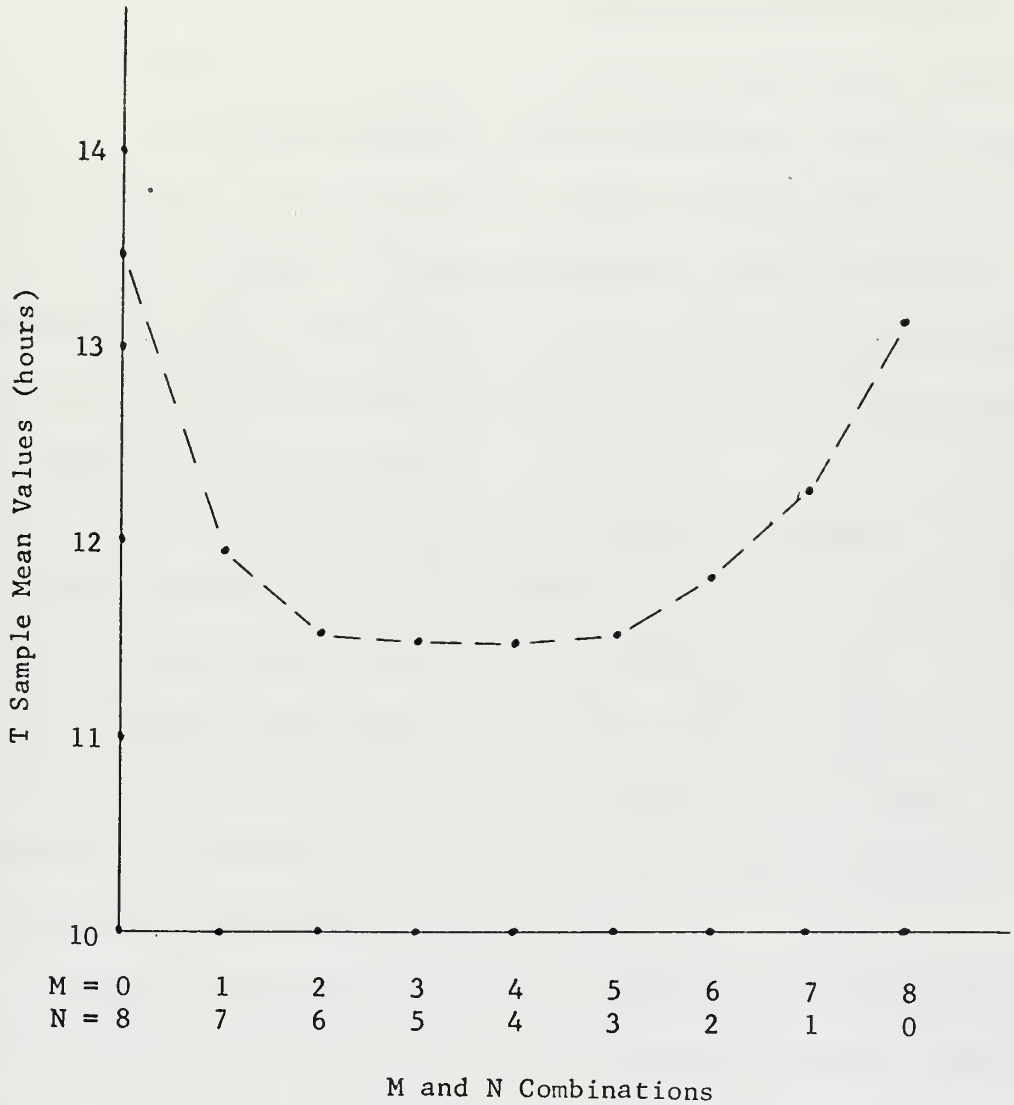
Total Replenishment Time by AO vs
Total Replenishment Time by AE when M = 5 and N = 3

FIGURE 6



Frequency Histogram of Typical T Observations
when M = 3 and N = 5

FIGURE 7



Relationship between T Sample Mean and Feasible M and N Combinations

FIGURE 8

necessary gamma distribution parameters k and λ, denoted $\hat{k}$ and $\hat{\lambda}$, were obtained by the method of moments as shown in Appendix B. The basic null hypothesis to be tested would be that the T realizations for a particular M and N combination came from a gamma distribution with parameters equal to $\hat{k}$ and $\hat{\lambda}$. The chi-square method was choosen for testing goodness of fit. Despite the generally acceptable results, eight out of nine null hypotheses being acceptable at the 0.01 level of significance, it was evident that better results could be achieved. By studying superimposed frequency histograms of the observed T realizations and the expected T realizations under the gamma distribution of the null hypotheses for both the accepted and rejected cases, two observations could be made:

a. the peak ordinate value for the number of observed T realizations was higher than that of the peak ordinate value for the number of expected T realizations for all M and N combinations; and

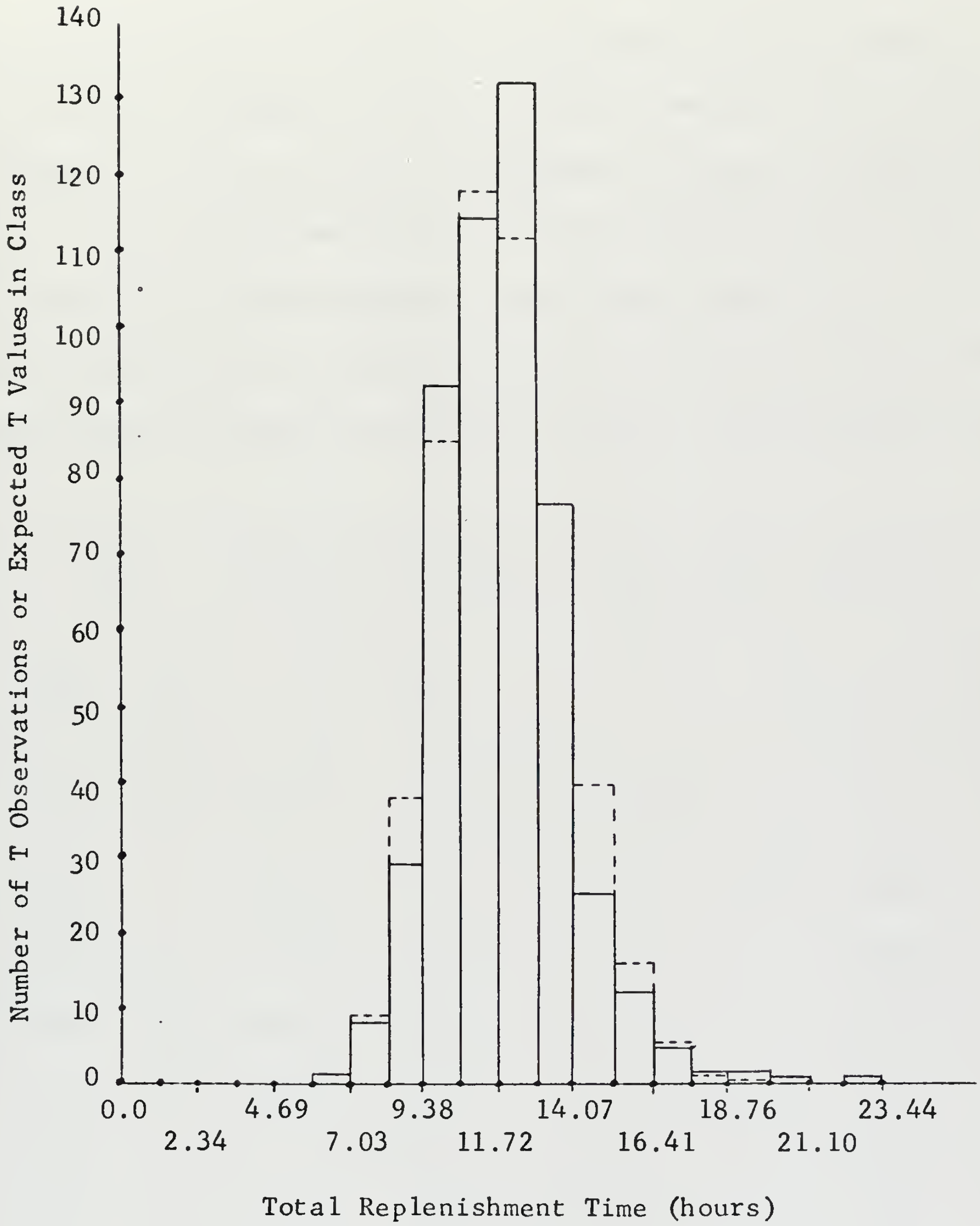
b. the mode, the absissca for the peak ordinate value, for the number of observed T realizations either occurred to the right of or approximately coincided with the mode for the number of expected T realizations for all M and N combinations.

A typical superimposed frequency histogram is shown in Figure 9.

From these observations, the following generalization could be made:

the true density function of T has a maximum value that occurs above and to the right of a gamma density function with parameters $\hat{k}$ and $\hat{\lambda}$.

Thus it was decided to use a method of estimating gamma parameters that utilized the maximum value of the gamma density function. This



Solid line: number of T observations
Dashed line: number of expected T values

<u>Note</u>: Number of expected T values calculated using gamma distribution with parameters estimated by method of moments

Superimposed Frequency Histograms
of Observed and Expected T Values when M = 6 and N = 2

FIGURE 9

method, termed the peak value method, is developed in Appendix B. Estimates of k and λ, as determined by this method, are denoted k* and λ*, respectively. Success was readily achieved by this method. The basic null hypothesis, that T for a particular M and N combination was distributed by a gamma distribution with parameters k* and λ*, was acceptable at the 0.01 level of significance by the chi-square method for all nine M and N combinations. The results, including those obtained by the method of moments, are shown in Table III. It could be concluded that the distribution function of T could be successfully approximated by a gamma distribution function with parameters k* and λ*.

RESULTS OF METHOD OF MOMENTS AND PEAK VALUE ESTIMATION						
M/N	$\bar{T}$	S_T^2	$\hat{k}$	$\hat{\lambda}$	k*	λ*
8/0 #	13.12	4.24	40.58	3.09	46.19	3.52
7/1	12.28	4.08	36.95	3.01	41.32	3.36
6/2	11.83	3.76	37.24	3.15	44.63	3.77
5/3	11.52	3.18	41.76	3.63	48.66	4.23
4/4	11.49	3.17	41.67	3.63	47.70	4.15
3/5	11.50	3.19	41.46	3.60	46.34	4.03
2/6	11.53	3.22	41.28	3.58	46.60	4.04
1/7	11.95	3.97	35.91	3.00	45.06	3.77
0/8	13.46	4.71	38.43	2.86	46.93	3.49

#: M and N combination that resulted in rejection of null hypothesis under method of moments testing

TABLE III

APPENDIX A

ILLUSTRATION OF PROBLEM WHEN L = 3 WITH M = 2 AND N = 1

The elements of this particular problem are presented below in the context of Section IV:

a. Considering the feasible paths, this problem involves random variables of the form:

$$\begin{aligned}
T_A &= X_1 + X_2 + X_3 \\
T_B^1 &= X_3 + Y_1 \\
T_C &= Y_1 + Y_2 + Y_3 \\
T_D^0 &= X_1 + Y_2 + Y_3 \\
T_D^1 &= X_1 + X_2 + Y_3.
\end{aligned}$$

The distribution function of T can be written

$$P(T \leq t) = P(X_1+X_2+X_3 \leq t,\ X_3+Y_1 \leq t,\ Y_1+Y_2+Y_3 \leq t,\ X_1+Y_2+Y_3 \leq t,\ X_1+X_2+Y_3 \leq t).$$

b. After the addition of the appropriate artificial random variable, the problem involves random variables of the form:

$$\begin{aligned}
T_1 &= X_1 + X_2 + X_3 \\
T_2 &= X_2 + X_3 \\
T_3 &= X_3 + Y_1 \\
T_4 &= Y_1 + Y_2 + Y_3 \\
T_5 &= X_1 + Y_2 + Y_3 \\
T_6 &= X_1 + X_2 + Y_3.
\end{aligned}$$

The distribution function of T can be written

$$P(T \leq t) = P(T_i \leq t \text{ for } i = 1,2,\ldots,6).$$

c. After the transformation from the T_n to the V_m, the problem involves random variables of the form:

$$\begin{aligned}
V_1 &= X_1 \\
V_2 &= X_2 - Y_1 \\
V_3 &= X_3 - Y_2 - Y_3 \\
V_4 &= -X_1 + Y_1 \\
V_5 &= -X_2 + Y_2 \\
V_6 &= X_1 + X_2 + Y_3.
\end{aligned}$$

The distribution function of T can be written

$$P(T \leq t) = P \sum_{m=i}^{6} V_m \quad t \text{ for } i = 1,2,\ldots,6).$$

d. Considering the inverse transformations from the V_m to the X_i and the Y_j, the problem involves the forms:

$$\begin{aligned}
X_1 &= V_1 \\
X_2 &= V_1 + V_2 + V_4 \\
X_3 &= -V_1 + V_3 + V_5 + V_6 \\
Y_1 &= V_1 + V_4 \\
Y_2 &= V_1 + V_2 + V_4 + V_5 \\
Y_3 &= -2V_1 - V_2 - V_4 + V_6.
\end{aligned}$$

e. The joint density function of the V_m, as wholly expressed in terms of the V_m, can be written

$$g(v_1,v_2,v_3,v_4,v_5,v_6) = \begin{cases} f_1(v_1)f_2(v_1+v_2+v_4)f_3(-v_1+v_3+v_5+v_6) \times \\ g_1(v_1+v_4)g_2(v_1+v_2+v_4+v_5)g_3(-2v_1-v_2-v_4+v_6) \\ \text{if condition (1) below holds} \\ 0 \qquad \text{otherwise} \end{cases}$$

Condition (1) is as follows

$$\begin{aligned} V_1 &> 0 \\ V_1 + V_2 + V_4 &> 0 \\ V_3 + V_5 + V_6 &> V_1 \\ V_1 + V_4 &> 0 \\ V_1 + V_2 + V_4 + V_5 &> 0 \\ V_6 &> 2V_1 + V_2 + V_4 \end{aligned}$$

APPENDIX B

RELEVANT THEORETICAL AND STATISTICAL FORMS

1. The Gamma Distribution

The gamma distribution function is given by

$$F(x) = \int_0^x \frac{\lambda^k x^{k-1} e^{-\lambda x}}{\Gamma(k)} dx \qquad \text{for } x > 0 \text{ and } k > 0,$$

where $\Gamma(k) = \int_0^\infty x^{k-1} e^{-x} dx$. Let μ, m, and σ^2 denote the mean, the mode, and the variance of the gamma distribution, then

$$\mu = k/\lambda;$$

$$m = (k-1)/\lambda; \text{ and}$$

$$\sigma^2 = k/\lambda^2.$$

If k is a positive integer, then the resulting gamma distribution is also termed an Erlang distribution. The Erlang distribution can be shown to be that for the sum of k identically distributed independent random variables from an exponential distribution with parameter λ. If k is of the form n/2 with n a positive integer and λ equals 1/2, then the resulting gamma distribution is also termed a chi-square distribution.

2. Determination of Erlang Random Variables by the Inverse Transformation Method

Let F(x) denote the distribution function for a random variable X and let r denote a random number from the uniform distribution on the interval 0 to 1. If $r = F(x)$, then $x = F(r)^{-1}$, where $F(r)^{-1}$ denotes the inverse transformation of F(x).* If X_i is exponentially distributed

*The transformation of a random variable X into a uniformly distributed random variable $r = F(x)$ is a well known result in probability theory.

with parameter λ, then

$$F(x_i) = 1-e^{-\lambda x_i}; \text{ or if } F(x_i) = r_i,$$

then $$r_i = 1-e^{-\lambda x_i};$$

thus $e^{-\lambda x_i} = 1 - r_i$; but if r_i is a realization from the uniform distribution on the interval 0 to 1, then $1 - r_i = r_i'$ is also a realization on the same interval. Thus this may be rewritten

$$\therefore \quad e^{-\lambda x_i} = r_i'; \text{ solving now for } x_i,$$

$$x_i = - 1/\lambda \ln r_i'.$$

If $X = \sum_{i=1}^{k} X_i$, then X has an Erlang distribution with parameters k and λ; thus

$$X = - 1/\lambda \sum_{i=1}^{k} \ln r_i'; \text{ which may be rewritten}$$

$$X = - 1/\lambda \ln\left(\prod_{i=1}^{k} r_i'\right), \text{ where}$$

$$F(r)^{-1} = - 1/\lambda \ln\left(\prod_{i=1}^{k} r_i'\right) \text{ with } r = \prod_{i=1}^{k} r_i'.$$

3. Test for Goodness of Fit by Chi-square Criteria

Let x_i be a realization from the sampled population; let 0_j denote the number of realizations that fall into class j for j = 1,2,...,n, where n is equal to some arbitrary number of subdivisions of the range of the assumed distribution of X_i; let E_j denote the expected number of realizations in class j under the assumed distribution; then the statistic

$$X = \sum_{i=1}^{n} \frac{(E_j - 0_j)^2}{E_j}$$

has an approximate chi-squared distribution with parameter (n-m-1)/2, where m denotes the number of assumed distribution parameters estimated

with sample statistics. n-m-1 is called the degree of freedom associated with X; for a choosen significance level the value of X can be compared with the appropriate table value such as found in Appendix 4 of [5]; if X is greater than the table value, then the null hypotheses that the x_i came from the assumed distribution is rejected; it is accepted otherwise.

4. Estimation of Gamma Parameters by the Method of Moments

Let T be a random variable from a gamma distribution with parameters k and λ. Let μ_T denote the expected value of T, $\bar{T}$ denote the T sample mean value, σ_T^2 denote the variance of T, and S_T^2 denote the T sample variance as calculated by $1/(n-1) \sum_{i=1}^{n} (T_i-\bar{T})^2$, where n is the number of T sample values. Since

$$\mu_T = k/\lambda, \text{ and}$$

$$\sigma_T^2 = k/\lambda^2,$$

it can be shown that $\hat{k}$ and $\hat{\lambda}$, estimates of k and λ, respectively, can be calculated by

$$\hat{k} = \bar{T}^2/S_T^2 \quad \text{and}$$

$$\hat{\lambda} = \bar{T}/S_T^2.$$

if the mean μ and variance σ_T^2 are respectively estimated by the sample mean and variance.

5. Estimation of Gamma Parameters by the Peak Value Method

Let T be a random variable from a gamma distribution with parameters k and λ. Let μ_T denote the expected value of T, $\bar{T}$ denote the T sample mean value, p denote the maximum value, i.e., the peak, of the relevant gamma density function, and m denote the mode of the distribution of T, i.e., the T value at which p occurs. Let $\hat{\mu}_T$, $\hat{p}$, k* and λ* denote

estimates of μ_T, p, k and λ, respectively. Since p occurs at m and since $m = (k-1)/\lambda$, p can be evaluated as

$$p = \frac{(k-1)^{k-1} e^{1-k}}{\Gamma(k)}.$$

Now by applying an approximation formula for $\Gamma(k)$ as found in [6], i.e.,

$$\Gamma(k) = \sqrt{2\pi}\, e^{-k} k^{k-1/2} \text{ for k large and not an integer,}$$

p can be approximated as

$$p \approx \frac{\lambda e}{\sqrt{2\pi k}} (1-1/k)^{k-1}.$$

Since for large k, $(1-1/k)^{k-1}$ can be approximated by e^{-1}, p can be approximated as

$$(1) \qquad p \approx \frac{\lambda}{\sqrt{2\pi k}} .$$

To estimate k and λ, $\hat{\mu}_T$ and $\hat{p}$ will be used. Since $\mu_T = k/\lambda$ and since p can be approximated by formula (1), by combining these equations and solving for k, it is easily seen that

$$k \approx 2\pi p^2 \mu_T^2.$$

Now, knowing an approximation for k, an approximation of λ can be obtained from $\lambda = k/\mu_T$. Now, by letting $\hat{\mu}_T = \overline{T}$ and $\hat{p} = \max 0_j/nw$, where $\max 0_j$ is the number of T realizations in the modal observation class of the T sample frequency histogram, n is the number of T realizations in the sample, and w is the width of the modal class in the frequency histogram, k* and λ* can then be evaluated as

$$k^* = 2\pi\hat{p}^2\overline{T}^2$$

$$\lambda^* = k^*/\overline{T}.$$

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13. ABSTRACT

An analytical approach to the problem of replenishing task forces at sea is investigated, using a random walk model. The type of replenishment operations considered consists of two replenishment ships of different types, each replenishing L combatant ships of different types. The combatant ships are initially distributed into queues of length M and N between the two replenishment ships. As each combatant ship completes its first replenishment, it then enters the queue of the other replenishment ship. Thus each combatant ship is replenished by both replenishment ships. Replenishment times for the combatant ships are assumed to be independent, positive valued random variables whose distributions are dependent upon both replenishment ship type and combatant ship type. The replenishment of the task force is completed whenever 2L replenishments, two for each combatant ship, are completed. The total replenishment time is shown to be the maximum of certain partial sums of the individual replenishment times. A general expression for the distribution function of the total replenishment time is sought by various methods of analysis. A computer simulation is used to gain a better understanding of the form of the distribution function of total replenishment time for a specific underway replenishment operation. Some examples are discussed.

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